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On the problem of rigid inclusions between two dissimilar elastic half-spaces with smooth surfaces

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Abstract

The surface displacements and contact stress induced by rigid inclusions between two dissimilar elastic half-spaces are considered. Previous studies limited to flat rigid inclusions have revealed that the contact stress on opposite sides of the inclusions are identical and that the surface displacements of the two half-spaces are proportional. The solution is rederived here in a simple fashion by using the Boussinesq solution, and the conclusions of previous studies are generalized to non-flat rigid inclusions. © 1999 Elsevier Science Ltd. All rights reserved.

In a recent paper Gladwell (1995) considered the problem of a rigid inclusion of uniform thickness and arbitrary planform compressed between two dissimilar, isotropic, linear elastic half-spaces. He formulated this problem in terms of the elastic displacement using the Papkovich–Neuber representation (Gladwell, 1980), and with extensive manipulations had shown that:

(i) the surface displacements of the two half-spaces u_{z1} and u_{z2} are related by

$$\vartheta_1 u_{z1} = \vartheta_2 u_{z2} \tag{1}$$

where ϑ_i is an elastic constant related to the shear modulus μ_i and Poisson's ratio v_i by $\vartheta_i = \mu_i/(1-v_i)$;

(ii) the contact stress on either side of the inclusion are identical;

(iii) the normal displacements in the region where the half-spaces are in contact are not only equal, but are separately zero.

These conclusions were also observed by Selvadurai (1994) who analyzed the problem for an axisymmetrical rigid inclusion of constant thickness (disk).

It will be shown that these three conclusions may be deduced in a much simpler fashion by using the Boussinesq solution for the surface displacements of an elastic half-space due to surface tractions (Johnson, 1985). Moreover, it will be shown that within the context of linear elasticity, these conclusions hold for any finite number of rigid inclusions of general shape. The shape of the

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inclusions is assumed to maintain a specific symmetry as explained in the following, but is otherwise arbitrary.

Let the position of a point in an elastic half-space be given by a Cartesian coordinate system where the z axis measures the distance from the surface. The normal displacement of the surface due to a normal force is given by (Johnson, 1985)

$$u_z(x,y) = \frac{P(x',y')}{2\pi\vartheta}\frac{1}{r}, \tag{2}$$

where P is the magnitude of the force and $r = \sqrt{(x-x')^2+(y-y')^2}$ is the distance from the point of action of the force. When a normal compressive stress $\sigma(x,y)$ is applied to the surface of the half-space, the normal surface displacement is given by the Boussinesq solution

$$u_z(x,y) = \frac{1}{\vartheta}\int_{-\infty}^{\infty}\frac{\sigma(x',y')}{r}\,\mathrm{d}x'\,\mathrm{d}y'. \tag{3}$$

Consider the well-posed problem of two identical half-spaces pressed together by a compressive stress $\sigma_{zz} = -\sigma_\infty$ at $z \to \pm\infty$, with a finite number of rigid inclusions positioned between them (Fig. 1). The elastic half-spaces are isotropic and linear elastic. They have an elastic modulus $\vartheta = \vartheta_1$ and have smooth surfaces (i.e., frictionless and cohesionless). The inclusions are of arbitrary planform and are symmetric with respect to the midplane $z = 0$. The distance $h_1(x,y)$ from the midplane to the lower surface of the inclusions, and the distance $h_2(x,y)$ from the midplane to the upper surface are prescribed by

$$h_1(x,y) = f(x,y), \tag{4a}$$

$$h_2(x,y) = h_1(x,y). \tag{4b}$$

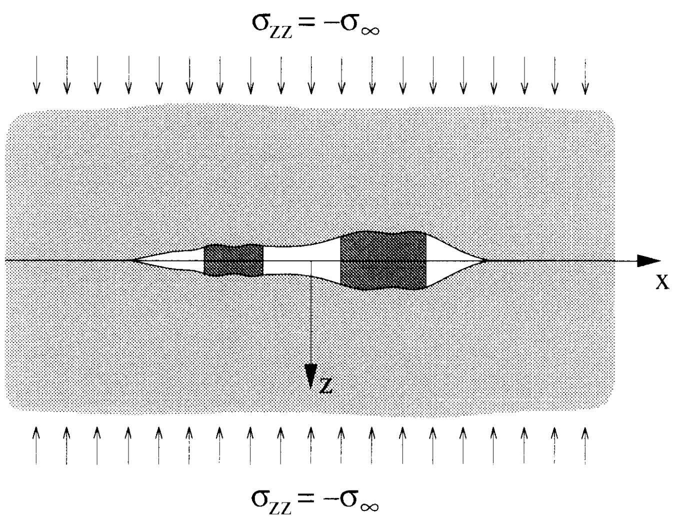

Fig. 1. Rigid inclusions between two identical elastic half-spaces.

To be consistent with Gladwell's notation, the surface displacement of each half-space is positive in its respective inward direction. The solution of this problem is symmetric and therefore the normal stress $\sigma_1(x, y)$ and normal displacement $u_{z1}(x, y)$ are identical at opposite points of the two surfaces. Moreover, the normal surface displacements are non-negative so that the two half-spaces do not overlap. Tensile normal stress is not possible so that the inclusions are not necessarily in full contact with the half-spaces. However, this does not affect the symmetry or the uniqueness of the solution. The normal surface displacement of either half-space may be described by the Boussinesq solution in the form

$$u_z(x, y) = \frac{1}{\vartheta_1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\sigma(x', y') - \sigma_\infty}{r}\,\mathrm{d}x'\,\mathrm{d}y'. \tag{5}$$

where the reference term $-\sigma_\infty$ ensures that the surface displacement vanishes far away from the inclusions.

The surface stress $\sigma_1(x, y)$ of the previous problem is now applied to two dissimilar elastic half spaces with elastic moduli ϑ_1 and ϑ_2, respectively. The normal surface displacement of the first half-space is given by eqn (5) while the normal surface displacement of the second half-space is given by

$$u_{z2}(x, y) = \frac{\vartheta_1}{\vartheta_2}u_{z1}(x, y). \tag{6}$$

Uniqueness of the solution in linear elasticity ensures that the surface stress σ_1 and displacements u_{z1}, u_{z2}, are also the solution of the following equivalent problem. The two dissimilar half-spaces are pressed together by a compressive stress $\sigma_{zz} = -\sigma_\infty$ at $z \to \pm\infty$, with a finite number of rigid inclusions positioned between them (Fig. 2). The inclusions are of arbitrary planform and the distances from the midplane to their surfaces are proportional. Specifically, the distance $h_1(x, y)$

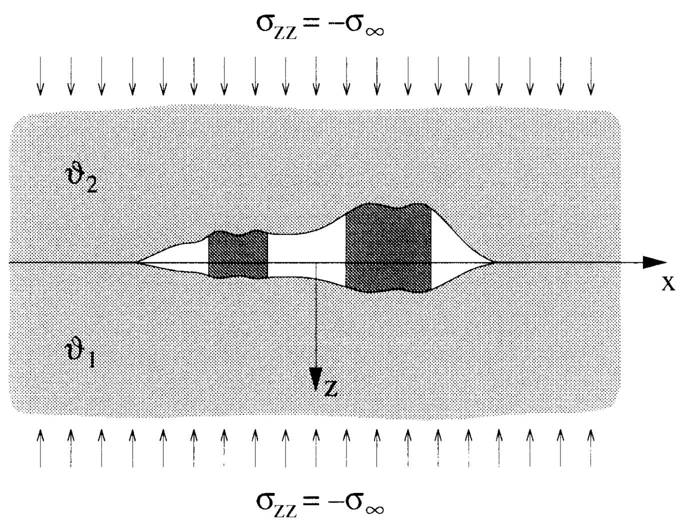

Fig. 2. Rigid inclusions between two dissimilar elastic half-spaces.

from the midplane $z = 0$ to the lower surface of the inclusions, and the distance $h_2(x, y)$ from the midplane to the upper surface are prescribed by

$$h_1(x, y) = f(x, y), \tag{7a}$$

$$h_2(x, y) = \frac{\vartheta_1}{\vartheta_2} f(x, y). \tag{7b}$$

Therefore, if the thickness of the inclusions satisfies the proportionality rule (7), then the three conclusions of Gladwell (1995) and Selvadurai (1994) must hold for rigid inclusions with uniform thickness. Within the context of linear elasticity where the stress is applied to the initial configuration with no reference to the current configuration, the three conclusions must also hold for rigid inclusions with varying thickness. Notice that the surfaces are smooth (i.e., frictionless and cohesionless) and therefore tensile normal stress at the surfaces is not possible. This means that the inclusions are not necessarily in full contact with the half-spaces and separation may occur. However, this does not affect the proportional symmetry of the surface displacements. The determination of inclusion thickness profiles that preclude such separation and the dependence of these thickness profiles on the inclusion planform remains an open problem.

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